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Addendum
On a generalization of the Gauss formula

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In this short note, we give a characterization of ZM-groups that uses the functions defined and studied in [M. Tărnăuceanu, A generalization of the Euler's totient function, *Asian-Eur. J. Math.* **8**(4) (2015) 1550087; M. Tărnăuceanu, On a generalization of the Gauss formula, *Asian-Eur. J. Math.* **10**(1) (2017) 1750008]. This leads to a proof of Conjecture 6 in [M. Tărnăuceanu, On a generalization of the Gauss formula, *Asian-Eur. J. Math.* **10**(1) (2017) 1750008].

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1. Introduction

The *Euler's totient function* (or, simply, the *totient function*) φ is one of the most famous functions in number theory. The totient $\varphi(n)$ of a positive integer n is defined to be the number of positive integers less than or equal to n, that are coprime to n. Note that there exist a lot of identities involving the totient function. One of them is the *Gauss formula*

$$\sum_{d|n} \varphi(d) = n, \quad \forall\, n \in \mathbb{N}^*. \tag{1}$$

In the last years, there has been a growing interest in extending arithmetical notions to finite groups (see e.g. [1–4, 6, 9]). Following this trend, a group theoretical generalization of φ has been introduced in [7]. It is founded on the remark that $\varphi(n)$ counts in fact the number of elements of order $n = \exp(C_n)$ in the cyclic group C_n. Consequently, it makes sense to define

$$\varphi(G) = |\{a \in G \mid o(a) = \exp(G)\}|$$

for any finite group G. It is obvious that

$$\varphi(C_n) = \varphi(n), \quad \text{for all } n \in \mathbb{N}^*,$$

and so a generalization of the classical totient function is obtained. We remark that for $G \cong C_n$ the Gauss formula (1) can be rewritten as

$$S(G) = |G|, \tag{2}$$

where

$$S(G) = \sum_{H \leq G} \varphi(H).$$

The class of finite groups G satisfying (2) has been partially determined in [8]. We are now able to complete this study by proving the following theorem. Recall that a *ZM-group* is a finite group having all Sylow subgroups cyclic (see e.g. [5]).

Theorem 1.1. *Let G be a finite group. Then $S(G) \geq |G|$, and we have equality if and only if G is a ZM-group.*

In particular, since a ZM-group is nilpotent if and only if it is cyclic, we get the result in [8, Conjecture 6].

Corollary 1.2. *Let G be a finite nilpotent group. Then $S(G) \geq |G|$, and we have equality if and only if G is cyclic.*

2. Proof of the Main Result

We start with an easy but important lemma.

Lemma 2.1. *Let H be a ZM-group. Then $\varphi(H) \neq 0$ if and only if H is cyclic.*

Proof. The "if" part follows immediately. Conversely, suppose that $\varphi(H) \neq 0$. Let $|H| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes. Since the Sylow p_i-subgroups of H are cyclic, there exists an element of H of order $p_i^{\alpha_i}$. We infer that $\exp(H) = |H|$. It is now clear that the condition $\varphi(H) \neq 0$ implies the existence of an element of H of order equal to $|H|$. Consequently, H is cyclic. $\square$

Note that the above argument also demonstrates that for a ZM-group G, the exponent of G coincides with the order of G.

We are now able to prove Theorem 1.1.

Proof of Theorem 1.1. Let $C(G)$ be the poset of cyclic subgroups of G. For every divisor d of $|G|$ we denote by n_d the number of cyclic subgroups of order d of

G and by n'_d the number of elements of order d in G. Then we have

$$n'_d = n_d \varphi(d)$$

because a cyclic subgroup of order d contains $\varphi(d)$ elements of order d. One obtains

$$S(G) \geq \sum_{H \in C(G)} \varphi(H) = \sum_{H \in C(G)} \varphi(|H|) = \sum_{d \,|\, |G|} n_d \varphi(d) = \sum_{d \,|\, |G|} n'_d = |G|,$$

as desired.

Assume now that $S(G) = |G|$. Then $\varphi(H) = 0$ for all non-cyclic subgroups H of G. We observe that for any p-group P we have $\varphi(P) \neq 0$. Indeed, if P is a p-group then every element has order equal to a power of p and the exponent of P is simply the maximum order, which is certainly achieved. It follows that all Sylow subgroups of G are cyclic, that is G is a ZM-group.

Conversely, assume that G is a ZM-group. Then every subgroup H of G is also a ZM-group, and by Lemma 2.1 we infer that

$$S(G) = \sum_{H \leq G} \varphi(H) = \sum_{H \in C(G)} \varphi(H) = |G|,$$

completing the proof. $\square$

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